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A local formulation of lattice Wess-Zumino model with exact $\mathrm{U}(1)_R$ symmetry

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Abstract: A lattice Wess-Zumino model is formulated on the basis of Ginsparg-Wilson fermions. In perturbation theory, our formulation is equivalent to the formulation by Fujikawa and Ishibashi and by Fujikawa. Our formulation is, however, free from a singular nature of the latter formulation due to an additional auxiliary chiral supermultiplet on a lattice. The model posssesses an exact $\mathrm{U}(1)_R$ symmetry as a supersymmetric counterpart of the Lüscher lattice chiral U(1) symmetry. A restration of the supersymmetric Ward-Takahashi identity in the continuum limit is analyzed in renormalized perturbation theory. In the one-loop level, a supersymmetric continuum limit is ensured by suitably adjusting a coefficient of a single local term $\tilde{F}^*\tilde{F}$. The non-renormalization theorem holds to this order of perturbation theory. In higher orders, on the other hand, coefficents of local terms with dimension ≤ 4 that are consistent with the $\mathrm{U}(1)_R$ symmetry have to be adjusted for a supersymmetric continuum limit. The origin of this complexicity in higher-order loops is clarified on the basis of the Reisz power counting theorem. Therefore, from a view point of supersymmetry, the present formulation is not quite better than a lattice Wess-Zumino model formulated by using Wilson fermions, although a number of coefficients which require adjustment is much less due to the exact $\mathrm{U}(1)_R$ symmetry. We also comment on an exact non-linear fermionic symmetry which corresponds to the one studied by Bonini and Feo; an existence of this exact symmetry itself does not imply a restoration of supersymmetry in the continuum limit without any adjustment of parameters.

Keywords: Renormalization Regularization and Renormalons, Lattice Quantum Field Theory, Global Symmetries, Supersymmetric Effective Theories.

Contents

1. Introduction

There has been a renewed interest on non-perturbative formulation of supersymmetric theories via a spacetime lattice [1]–[9] in these several years [10]–[21] (for a recent review with a complete list of references, see ref. [22]). One major idea in these recent studies is to keep a part of the supersymmetry algebra manifest and infer that this exact symmetry is strong enough to ensure a fully supersymmetric continuum limit without any (or with a small number of) adjustment of parameters [10, 15, 16, 19]. This general strategy, which is also common to some of past attempts [2], has achieved fair success, typically for lower dimensional supersymmetric theories with an extended supersymmetry (besides a potential problem of positivity of the measure). An extended supersymmetry allows a sub-algebra which is consistent with a lattice construction and, due to the lower-ness of dimensionality, the number of relevant operators, which potentially violate supersymmetry in the continuum limit, is small. So often a supersymmetric continuum limit is achieved without any adjustment of parameters.

Another kind of approaches is to abandon a manifest supersymmetry of a lattice model from the onset and achieve a supersymmetric continuum limit by adjusting parameters in the model. This is an approach advocated in refs. [3]–[5] and [7] and this has been, in our opinion, only a realistic approach to date for $N = 1$ supersymmetric theories in

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4 dimensions. Here, again, some exact global symmetries on a lattice can be useful [8, 9] to reduce the number of parameters which require adjustment. For numerical simulations along this kind of approaches, see ref. [23].[1]

In this paper, we adopt the latter attitude and study a lattice formulation of the 4 dimensional $N = 1$ supersymmetric Wess-Zumino model [24]. The Wess-Zumino model is asymptotic non-free and thus the continuum limit, as a fundamental theory, is expected to be trivial. Nevertheless, it is meaningful to consider the model as an effective theory with an ultraviolet cutoff. In a sense, this model is more difficult to formulate on a lattice than supersymmetric Yang-Mills theories because a quadratic divergence in mass terms of scalar fields is expected to be prohibited only with presence of an exact supersymmetry. It is thus deserve to study in its own right. We formulate the model on the basis of Ginsparg-Wilson fermions [25]–[29]. This kind of formulation has been pursued by Fujikawa and Ishibashi and by Fujikawa [11, 12, 14]. In our notation, their formulation is expressed as

$$\begin{aligned} S = a^4 \sum_x \bigg\{ & \frac{1}{2}\chi^T C \left(1 - \frac{1}{2}aD_2\right)^{-1} D_1\chi - \frac{2}{a}\phi^* D_2\phi + F^* \left(1 - \frac{1}{2}aD_2\right)^{-1} F + \\ & + \frac{1}{2}m\chi^T CP_+\chi + \frac{1}{2}m^*\chi^T CP_-\chi + mF\phi + m^*F^*\phi^* + \\ & + g\chi^T C\phi P_+\chi + gF\phi^2 + g^*\chi^T C\phi^* P_-\chi + g^*F^*\phi^{*2} \bigg\}, \end{aligned} \tag{1.1}$$

where (χ, ϕ, F) denote the chiral multiplet of the Wess-Zumino model and D_1 and D_2 are lattice difference operators which will be defined below. In ref. [11], explicit perturbative calculations in one-loop order were carried out and it was found that, in the one-loop level, effects of supersymmetry breaking in the model appear only in wave function renormalization factors of the chiral multiplet, thus the violation of supersymmetry is rather moderate in the one-loop level.

One can carry out perturbative calculations on the basis of the action (1.1) without any problem. However, as pointed out in ref. [12] (see also ref. [30]), the action itself is singular because the operator $1 - (1/2)aD_2$ always has zero modes. This also implies that the kinetic operator in eq. (1.1) is non-local. Thus the meaning of the model in a non-perturbative level is not clear.

In this paper, we first formulate a *non-singular* local lattice action for the Wess-Zumino model which is, in perturbation theory, equivalent to the formulation based on the action (1.1). This is achieved by introducing a non-dynamical auxiliary chiral multiplet on a lattice which decouples in the continuum limit. Due to the Ginsparg-Wilson relation, when $m = 0$, our model possesses a lattice analogue of the $\mathrm{U}(1)_R$ symmetry which is supersymmetric counterpart of the Lüscher lattice chiral U(1) symmetry [29].[2] This is the symmetry that was pointed out for a free theory in ref. [9]. These are contents of section 2.

[1] There exist more ambitious approaches which aim an exact full supersymmetry on a lattice [6, 13, 20]. Another interesting observation is that a supersymmetric continuum theory is automatically restored if a convergence behavior of Feynman integrals in a lattice model is moderate enough [14, 17].

[2] In the action (1.1), this $\mathrm{U}(1)_R$ symmetry is trivially realized as $\delta_\alpha\chi = +i\alpha\gamma_5\chi$, $\delta_\alpha\phi = -2i\alpha\phi$ and $\delta_\alpha F = +4i\alpha F$. See section 2.

Next, we study a restoration of supersymmetry in the continuum limit by using a lattice version of the Ward-Takahashi identity. We carry out an explicit one-loop evaluation of the supersymmetry breaking term in the Ward-Takahashi identity and observe that effects of supersymmetry breaking in the present model appear only in the wave function renormalization of the auxiliary field $\tilde{F}$ in the continuum limit (in the one-loop level). We then present a general argument for higher loop contributions of supersymmetry breaking in renormalized perturbation theory. Unfortunately, the general power counting argument which is based on the Reisz theorem [31, 32] indicates that all supersymmetric non-invariant local terms with the mass dimension ≤ 4 are radiatively induced, unless a term is forbidden by the $\mathrm{U}(1)_R$ global symmetry that is manifest in our formulation. We clarify the reason why the one-loop result is so simple and higher loop corrections are expected to be destructive. In terms of the power counting theorem, the supersymmetry breaking term, which is a consequence of a violation of the Leibniz rule on the lattice, behaves as a non-derivative coupling in one-loop diagrams while it behaves as a derivative coupling in higher loop diagrams. This peculiar behavior of the supersymmetry breaking term makes the situation in higher loop diagrams involved. As a conclusion, from a view of supersymmetry restoration, our formulation is not quite better than the formulation based on the Wilson fermion [3], although some of super non-invariant local terms are prohibited by the exact $\mathrm{U}(1)_R$ symmetry. (section 3)

In the final part of this paper, we will comment on an exact non-linear fermionic symmetry in our formulation which corresponds to the symmetry recently studied in ref. [21] in the context of the Fujikawa-Ishibashi formulation. This symmetry is nothing but the "lattice supersymmetry" utilized in ref. [7] for 2 dimensional Wess-Zumino model. As noted in ref. [7] and as indicated from the results of ref. [11] and of ours, a presence of this symmetry itself does not imply an automatic restoration of supersymmetry in the continuum limit without any adjustment of parameters (although the non-linear symmetry reduces to the standard supersymmetry in the classical continuum limit). We clarify this point.

Throughout this paper, the lattice spacing will be denoted by a.

2. The model

2.1 Action and its symmetries

Our staring point is the chiral invariant lattice Yukawa model of ref. [29]:

$$S = a^4 \sum_x \Big\{ \overline{\psi} D \psi - \frac{2}{a} \overline{\Psi} \Psi + 2g(\overline{\psi} + \overline{\Psi}) \Big(\phi + \frac{m}{2g} \Big) P_+ (\psi + \Psi) + \\ + 2g^* (\overline{\psi} + \overline{\Psi}) \left(\phi^* + \frac{m^*}{2g^*} \right) P_- (\psi + \Psi) \Big\}, \tag{2.1}$$

where the field Ψ is a non-dynamical auxiliary fermionic field and $P_\pm = (1 \pm \gamma_5)/2$. In this expression, we have shifted the scalar field as $\phi \to \phi + m/(2g)$ to generate mass terms for fermions. As the lattice Dirac operator D, we adopt the overlap-Dirac operator [26, 27] defined by[3]

$$D=\frac{1}{2}\{1-A(A^{\dagger}A)^{-1/2}\}\,,\qquad A=1-aD_{\mathrm{w}}\,,\qquad D_{\mathrm{w}}=\frac{1}{2}\{\gamma_\mu(\partial_\mu^*+\partial_\mu)-a\partial_\mu^*\partial_\mu\}\,,\quad (2.2)$$

which obeys the Ginsparg-Wilson relation $\gamma_5 D + D\gamma_5 = aD\gamma_5 D$ [25]. Thanks to this relation, the action with $m=0$ is invariant under a lattice chiral transformation of the following form [29]

$$\begin{aligned}
\delta_\alpha\psi &= i\alpha\gamma_5\left(1-\frac{1}{2}aD\right)\psi+i\alpha\gamma_5\Psi\,, &\qquad \delta_\alpha\Psi &= i\alpha\gamma_5\frac{1}{2}aD\psi\,,\\
\delta_\alpha\overline{\psi} &= i\alpha\overline{\psi}\left(1-\frac{1}{2}aD\right)\gamma_5+i\alpha\overline{\Psi}\gamma_5\,, &\qquad \delta_\alpha\overline{\Psi} &= i\alpha\overline{\psi}\frac{1}{2}aD\gamma_5\,,\\
\delta_\alpha\phi &= -2i\alpha\phi\,, &\qquad \delta_\alpha\phi^* &= 2i\alpha\phi^*\,,
\end{aligned}\qquad (2.3)$$

where α is an infinitesimal real parameter. This transformation is designed so that a sum of fields, say $\psi+\Psi$, transforms in a standard way, $\delta_\alpha(\psi+\Psi)=i\alpha\gamma_5(\psi+\Psi)$. Thus a breaking of this chiral symmetry due to the presence of mass terms has a simple structure as in the continuum theory. The auxiliary fermion Ψ is introduced to make a chiral transformation of this standard form and the Ginsparg-Wilson relation (which implies a non-standard chiral property of the lattice Dirac operator D) compatible.

To define the Wess-Zumino model, we need to reduce degrees of freedom of the Dirac field ψ to the Majorana one. Since the chiral projectors $P_\pm=(1\pm\gamma_5)/2$ in the Yukawa interaction term are ordinary ones, the Majorana reduction (see ref. [11]) can be applied straightforwardly. Namely, we make substitutions[4]

$$\begin{aligned}
\psi &= \frac{(\chi+i\eta)}{\sqrt{2}}\,, &\qquad \overline{\psi} &= \frac{(\chi^T C-i\eta^T C)}{\sqrt{2}}\,,\\
\Psi &= \frac{(X+iY)}{\sqrt{2}}\,, &\qquad \overline{\Psi} &= \frac{(X^T C-iY^T C)}{\sqrt{2}}\,,
\end{aligned}\qquad (2.4)$$

in the action. Noting relations

$$(CD)^T=-CD,\qquad (CP_\pm)=-CP_\pm\,,\qquad (2.5)$$

we find that the action decomposes into two independent systems. By taking only terms including χ and X, we have[5]

$$\begin{aligned}
S = a^4\sum_x\Bigg\{&\frac{1}{2}\chi^T CD\chi-\frac{1}{a}X^T CX+g(\chi^T+X^T)C\left(\phi+\frac{m}{2g}\right)P_+(\chi+X)\\
&+g^*(\chi^T+X^T)C\left(\phi^*+\frac{m^*}{2g^*}\right)P_-(\chi+X)\Bigg\}\,.
\end{aligned}\qquad (2.6)$$

[3] $\partial_\mu f(x)=\{f(x+a\hat{\mu})-f(x)\}/a$ and $\partial_\mu^* f(x)=\{f(x)-f(x-a\hat{\mu})\}/a$ are the forward and backward difference operators, respectively.

[4] C is the charge conjugation matrix which satisfies $C\gamma_\mu C^{-1}=-\gamma_\mu^T$, $C\gamma_5 C^{-1}=\gamma_5^T$, $C^\dagger C=1$ and $C^T=-C$.

[5] This Majorana reduction, in a level of the functional integral, corresponds to the prescription of ref. [33].

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When $m = 0$, the action is still invariant under the chiral transformation

$$
\begin{aligned}
\delta_\alpha \chi &= i\alpha\gamma_5 \left(1 - \frac{1}{2}aD\right)\chi + i\alpha\gamma_5 X\,, \qquad \delta_\alpha X = i\alpha\gamma_5 \frac{1}{2}aD\chi\,, \\
\delta_\alpha \phi &= -2i\alpha\phi\,, \qquad \delta_\alpha \phi^* = 2i\alpha\phi^*\,.
\end{aligned}
\tag{2.7}
$$

Eq. (2.6) provides a part of our lattice Wess-Zumino model, kinetic terms of fermions and Yukawa interaction terms.

We next introduce bosonic superpartners of fermion fields, (ϕ, F) and $(\Phi, \mathcal{F})$, and seek an appropriate free action which is invariant under a certain "lattice supersymmetry". As the form of this fermionic transformation, we postulate

$$
\begin{aligned}
\delta_\epsilon \chi &= -\sqrt{2}P_+(D_1\phi + F)\epsilon - \sqrt{2}P_-(D_1\phi^* + F^*)\epsilon\,, \\
\delta_\epsilon \phi &= \sqrt{2}\epsilon^T C P_+ \chi\,, \qquad \delta_\epsilon \phi^* = \sqrt{2}\epsilon^T C P_- \chi\,, \\
\delta_\epsilon F &= \sqrt{2}\epsilon^T C D_1 P_+ \chi\,, \qquad \delta_\epsilon F^* = \sqrt{2}\epsilon^T C D_1 P_- \chi\,,
\end{aligned}
\tag{2.8}
$$

and

$$
\begin{aligned}
\delta_\epsilon X &= -\sqrt{2}P_+(D_1\Phi + \mathcal{F})\epsilon - \sqrt{2}P_-(D_1\Phi^* + \mathcal{F}^*)\epsilon\,, \\
\delta_\epsilon \Phi &= \sqrt{2}\epsilon^T C P_+ X\,, \qquad \delta_\epsilon \Phi^* = \sqrt{2}\epsilon^T C P_- X\,, \\
\delta_\epsilon \mathcal{F} &= \sqrt{2}\epsilon^T C D_1 P_+ X\,, \qquad \delta_\epsilon \mathcal{F}^* = \sqrt{2}\epsilon^T C D_1 P_- X\,.
\end{aligned}
\tag{2.9}
$$

In this expression, ϵ is a 4 component Grassmann parameter and we have used a decomposition of the Dirac operator, $D = D_1 + D_2$, where

$$
D_1 = \frac{1}{2}\gamma_\mu(\partial_\mu^* + \partial_\mu)(A^\dagger A)^{-1/2}\,, \qquad D_2 = \frac{1}{a}\left\{1 - (1 + \frac{1}{2}a^2\partial_\mu^*\partial_\mu)(A^\dagger A)^{-1/2}\right\}. \tag{2.10}
$$

Note that with respect to spinor space, D_1 and D_2 have different structures. In particular, we have $\{\gamma_5, D_1\} = 0$ and $[\gamma_5, D_2] = 0$. In terms of this decomposition, the Ginsparg-Wilson relation is expressed as

$$
2D_2 = a(-D_1^2 + D_2^2)\,, \tag{2.11}
$$

and as a consequence, we have relations

$$
\gamma_5\left(1 - \frac{1}{2}aD\right)\gamma_5\left(1 - \frac{1}{2}aD\right) = 1 - \frac{1}{2}aD_2\,, \qquad \gamma_5\left(1 - \frac{1}{2}aD\right)\gamma_5 D = D_1\,, \tag{2.12}
$$

which will frequently be used below. It is also understood that the 4×4 identity matrix in operators D_1^2 and D_2 is omitted when these operators are acting on bosonic fields. It is then straightforward to see that the following free action is invariant under eqs. (2.8) and (2.9):

$$
\begin{aligned}
S_0 = a^4 \sum_x \Big\{ &\frac{1}{2}\chi^T C D \chi + \phi^* D_1^2 \phi + F^* F + F D_2 \phi + F^* D_2 \phi^* - \\
&- \frac{1}{a}X^T C X - \frac{2}{a}(\mathcal{F}\Phi + \mathcal{F}^*\Phi^*) + \\
&+ \frac{1}{2}m\tilde{\chi}^T C P_+ \tilde{\chi} + \frac{1}{2}m^*\tilde{\chi}^T C P_- \tilde{\chi} + m\tilde{F}\tilde{\phi} + m^*\tilde{F}^*\tilde{\phi}^* \Big\}\,,
\end{aligned}
\tag{2.13}
$$

where we have introduced abbreviations

$$\tilde{\chi} = \chi + X, \qquad \tilde{\phi} = \phi + \Phi\,, \qquad \tilde{F} = F + \mathcal{F}\,. \tag{2.14}$$

The combinations (χ, ϕ, F) and $(X, \Phi, \mathcal{F})$ are regarded as chiral multiplet in the lattice model. In particular, we refer $(X, \Phi, \mathcal{F})$ to as the auxiliary chiral multiplet which is characteristic in the present lattice formulation.

We note that the free action S_0 with $m = 0$ possesses three types of U(1) symmetry [9]. The first is a rather trivial one acting only on bosonic fields and is defined by the transformation:

$$\begin{aligned} \delta_\alpha \chi &= 0\,, & \delta_\alpha X &= 0\,, \\ \delta_\alpha \phi &= i\alpha\phi\,, & \delta_\alpha \Phi &= i\alpha\Phi\,, \\ \delta_\alpha F &= -i\alpha F\,, & \delta_\alpha \mathcal{F} &= -i\alpha\mathcal{F}\,, \end{aligned} \tag{2.15}$$

where α is an infinitesimal real parameter. This remains the symmetry of S_0 even for $m \neq 0$. The second one is nothing but the Lüscher chiral symmetry, (2.7) with $\delta_\alpha\phi = 0$, $\delta_\alpha\Phi = 0$, $\delta_\alpha F = 0$ and $\delta_\alpha\mathcal{F} = 0$. Thirdly, somewhat surprisingly, the *bosonic* sector of S_0 with $m = 0$ possesses an analogous U(1) symmetry to eq. (2.7):

$$\begin{aligned} &\delta_\alpha \chi = 0\,, \qquad \delta_\alpha X = 0\,, \\ &\delta_\alpha \phi = +i\alpha\left\{\left(1 - \frac{1}{2}aD_2\right)\phi - \frac{1}{2}aF^*\right\} + i\alpha\Phi\,, \qquad \delta_\alpha\Phi = +i\alpha\left\{\frac{1}{2}aD_2\phi + \frac{1}{2}aF^*\right\}, \\ &\delta_\alpha F = +i\alpha\left\{\left(1 - \frac{1}{2}aD_2\right)F - \frac{1}{2}aD_1^2\phi^*\right\} + i\alpha\mathcal{F}\,, \qquad \delta_\alpha\mathcal{F} = +i\alpha\left\{\frac{1}{2}aD_2F + \frac{1}{2}aD_1^2\phi^*\right\}, \end{aligned} \tag{2.16}$$

due to the Ginsparg-Wilson relation. The lattice action S_0 is not invariant under a uniform rotation of the complex phase of bosonic fields, ϕ, F, Φ and $\mathcal{F}$, due to the presence of terms $FD_2\phi$ and $F^*D_2\phi^*$. The above provides a lattice counterpart of this uniform phase rotation of bosonic fields under which the free action S_0 with $m = 0$ is invariant. Using a linear combination of the above three U(1) symmetries, it is possible to define the $\mathrm{U}(1)_R$ symmetry [9] in the interacting system, as we will see below. It is worthwhile to note that a sum of transformations (2.15) and (2.16) takes the following simple form when acting on tilded variables:

$$\delta_\alpha\tilde{\chi} = +i\alpha\gamma_5\tilde{\chi}\,, \qquad \delta_\alpha\tilde{\phi} = +i\alpha\tilde{\phi}\,, \qquad \delta_\alpha\tilde{F} = +i\alpha\tilde{F}\,. \tag{2.17}$$

Next we postulate a form of the interaction term as

$$S_{\text{int.}} = a^4\sum_x\left\{g\tilde{\chi}^T C\tilde{\phi}P_+\tilde{\chi} + g\tilde{F}\tilde{\phi}^2 + g^*\tilde{\chi}^T C\tilde{\phi}^* P_-\tilde{\chi} + g^*\tilde{F}^*\tilde{\phi}^{*2}\right\}, \tag{2.18}$$

where we have defined interaction terms by taking tilded variables (2.14) as unit, because in this way we can relate our formulation to the Fujikawa-Ishibashi formulation. This way of construction of interaction terms is also a natural generalization of the Yukawa interaction in eq. (2.1). We then find that the full action $S = S_0 + S_{\text{int.}}$ is "almost" invariant under the fermionic transformations (2.8) and (2.9). In fact, after some algebra using the Fierz identity,[6] we obtain

$$\begin{aligned}\delta_\epsilon S &= -a^4\sum_x \tilde\chi^T C\sqrt{2}\left\{gP_+(2\tilde\phi D_1\tilde\phi - D_1\tilde\phi^2)\epsilon + g^*P_-(2\tilde\phi^* D_1\tilde\phi^* - D_1\tilde\phi^{*2})\epsilon\right\}\\ &\equiv -a^4\sum_x \tilde\chi^T C\Delta L\epsilon\,. \end{aligned}\tag{2.19}$$

We emphasize that this breaking could vanish if the Leibniz rule was valid for the lattice difference operator D_1. In summary, the lattice action for the Wess-Zumino model

$$\begin{aligned}S = a^4\sum_x\Big\{&\frac{1}{2}\chi^T CD\chi + \phi^* D_1^2\phi + F^*F + FD_2\phi + F^*D_2\phi^* -\\ &-\frac{1}{a}X^T CX - \frac{2}{a}(\mathcal{F}\Phi + \mathcal{F}^*\Phi^*) +\\ &+\frac{1}{2}m\tilde\chi^T CP_+\tilde\chi + \frac{1}{2}m^*\tilde\chi^T CP_-\tilde\chi + m\tilde F\tilde\phi + m^*\tilde F^*\tilde\phi^* +\\ &+g\tilde\chi^T C\tilde\phi P_+\tilde\chi + g\tilde F\tilde\phi^2 + g^*\tilde\chi^T C\tilde\phi^* P_-\tilde\chi + g^*\tilde F^*\tilde\phi^{*2}\Big\}\,,\end{aligned}\tag{2.20}$$

is invariant under the lattice super transformation (2.8) and (2.9) up to the breaking term (2.19).

For the action (2.20), we can define two types of exact global "symmetries". The first is eq. (2.15) which yields on tilded variables

$$\delta_\alpha\tilde\chi = 0\,,\qquad \delta_\alpha\tilde\phi = +i\alpha\tilde\phi\,,\qquad \delta_\alpha\tilde F = -i\alpha\tilde F\,.\tag{2.21}$$

This is not a symmetry when $g \neq 0$, but may be regarded as a "symmetry" if we simultaneously rotate the coupling constant according to

$$\delta_\alpha g = -i\alpha g\,.\tag{2.22}$$

Another is a lattice counterpart of the $\mathrm{U}(1)_R$ symmetry which is given by a linear combination of the above three U(1) transformations:

$$\begin{aligned}\delta_\alpha\chi &= +i\alpha\gamma_5\left(1-\frac{1}{2}aD\right)\chi + i\alpha\gamma_5 X\,,\qquad \delta_\alpha X = +i\alpha\gamma_5\frac{1}{2}aD\chi\,,\\ \delta_\alpha\phi &= -3i\alpha\phi + i\alpha\left\{\left(1-\frac{1}{2}aD_2\right)\phi - \frac{1}{2}aF^*\right\} + i\alpha\Phi\,,\\ \delta_\alpha\Phi &= -3i\alpha\Phi + i\alpha\left\{\frac{1}{2}aD_2\phi + \frac{1}{2}aF^*\right\},\\ \delta_\alpha F &= +3i\alpha F + i\alpha\left\{\left(1-\frac{1}{2}aD_2\right)F - \frac{1}{2}aD_1^2\phi^*\right\} + i\alpha\mathcal{F}\,,\\ \delta_\alpha\mathcal{F} &= +3i\alpha\mathcal{F} + i\alpha\left\{\frac{1}{2}aD_2F + \frac{1}{2}aD_1^2\phi^*\right\}.\end{aligned}\tag{2.23}$$

On tilded variables, this $\mathrm{U}(1)_R$ transformation takes a simple form

$$\delta_\alpha\tilde\chi = +i\alpha\gamma_5\tilde\chi\,,\qquad \delta_\alpha\tilde\phi = -2i\alpha\tilde\phi\,,\qquad \delta_\alpha\tilde F = +4i\alpha\tilde F\,.\tag{2.24}$$

[6] We use the identity $(\chi^T C\chi)\chi = -(\chi^T C\gamma_5\chi)\gamma_5\chi$.

The action S with $m = 0$ is invariant under this transformation and this may also be regarded as a "symmetry" even for $m \neq 0$ if we transform the mass parameter according to

$$\delta_\alpha m = -2i\alpha m\,. \tag{2.25}$$

Eqs. (2.23)–(2.25) correspond to the $\mathrm{U}(1)_R$ transformation of the continuum Wess-Zumino model. In fact, if we define the $\mathrm{U}(1)_R$ transformation of the parameter of the super transformation as

$$\delta_\alpha \epsilon = -3i\alpha\gamma_5\epsilon\,, \tag{2.26}$$

then from eqs. (2.8) and (2.9) it can be confirmed that $[\delta_\epsilon, \delta_\alpha] = 0$ holds on all field variables. The above two "symmetries" play an important role when we consider a structure of radiative corrections in the present model.

In our non-singular local action (2.20) with interactions, the chiral $\mathrm{U}(1)_R$ symmetry is realized as an exact symmetry. The no-go theorem of ref. [30] on a chiral invariant Yukawa interaction of the Majorana fermion is evaded in our formulation due to an introduction of the auxiliary field(s). We further clarify this point in the next subsection.

2.2 Perturbative equivalence to the Fujikawa-Ishibashi formulation

In perturbation theory, the above system S is completely equivalent to a lattice Wess-Zumino model formulated in refs. [11, 12, 14], i.e., eq. (1.1). A *formal* way to see this equivalence is to rewrite the action S in favor of tilded variables and of the auxiliary multiplet $(X, \Phi, \mathcal{F})$:

$$\begin{aligned} S = a^4 \sum_x \bigg\{ &\frac{1}{2}(\tilde{\chi}^T - X^T)CD(\tilde{\chi} - X) + (\tilde{\phi}^* - \Phi^*)D_1^2(\tilde{\phi} - \Phi) + \\ &+ (\tilde{F}^* - \mathcal{F}^*)(\tilde{F} - \mathcal{F}) + (\tilde{F} - \mathcal{F})D_2(\tilde{\phi} - \Phi) + (\tilde{F}^* - \mathcal{F}^*)D_2(\tilde{\phi}^* - \Phi^*) - \\ &- \frac{1}{a}X^T CX - \frac{2}{a}(\mathcal{F}\Phi + \mathcal{F}^*\Phi^*) + \\ &+ \frac{1}{2}m\tilde{\chi}^T CP_+\tilde{\chi} + \frac{1}{2}m^*\tilde{\chi}^T CP_-\tilde{\chi} + m\tilde{F}\tilde{\phi} + m^*\tilde{F}^*\tilde{\phi}^* + \\ &+ g\tilde{\chi}^T C\tilde{\phi}P_+\tilde{\chi} + g\tilde{F}\tilde{\phi}^2 + g^*\tilde{\chi}^T C\tilde{\phi}^* P_-\tilde{\chi} + g^*\tilde{F}^*\tilde{\phi}^{*2} \bigg\}\,. \end{aligned} \tag{2.27}$$

If we perform integrations over the auxiliary chiral multiplet, X, Φ and $\mathcal{F}$, we have the effective action for tilded variables:

$$\begin{aligned} \tilde{S} = a^4 \sum_x \bigg\{ &\frac{1}{2}\tilde{\chi}^T C \left(1 - \frac{1}{2}aD_2\right)^{-1} D_1\tilde{\chi} - \frac{2}{a}\tilde{\phi}^* D_2\tilde{\phi} + \tilde{F}^* \left(1 - \frac{1}{2}aD_2\right)^{-1} \tilde{F} + \\ &+ \frac{1}{2}m\tilde{\chi}^T CP_+\tilde{\chi} + \frac{1}{2}m^*\tilde{\chi}^T CP_-\tilde{\chi} + m\tilde{F}\tilde{\phi} + m^*\tilde{F}^*\tilde{\phi}^* + \\ &+ g\tilde{\chi}^T C\tilde{\phi}P_+\tilde{\chi} + g\tilde{F}\tilde{\phi}^2 + g^*\tilde{\chi}^T C\tilde{\phi}^* P_-\tilde{\chi} + g^*\tilde{F}^*\tilde{\phi}^{*2} \bigg\}\,. \end{aligned} \tag{2.28}$$

This is, if we identify tilded variables as basic field variables, nothing but the action (1.1). Associated to the integration over the auxiliary chiral multiplet, we have

$$\mathrm{Pf}\left\{C\left(D - \frac{2}{a}\right)\right\} \det{}^{-1}\left\{\frac{2}{a}\left(D_2 - \frac{2}{a}\right)\right\}, \tag{2.29}$$

where the first factor comes from an integration over the 4 component fermionic spinor X and the second comes from an integration over complex bosonic scalars Φ and $\mathcal{F}$; the operator in the latter factor therefore does not contain 4×4 identity matrix in spinor space. In a formal sense, these two factors are cancelled out, because the relation

$$\gamma_5 \left(D - \frac{2}{a} \right) \gamma_5 \left(D - \frac{2}{a} \right) = -\frac{2}{a} \left(D_2 - \frac{2}{a} \right) \tag{2.30}$$

holds (recall eq. (2.12)) and thus we have

$$\det{}^2 \left\{ D - \frac{2}{a} \right\} = \det{}^4 \left\{ \frac{-2}{a} \left(D_2 - \frac{2}{a} \right) \right\}, \tag{2.31}$$

by noting the fact that the right hand side of eq. (2.30) contains the 4×4 identity matrix. Therefore, we see

$$\mathrm{Pf} \left\{ C \left(D - \frac{2}{a} \right) \right\} = \det{}^{1/2} C \det{}^{1/2} \left\{ D - \frac{2}{a} \right\} = \det{}^{1/2} C \det \left\{ \frac{-2}{a} \left(D_2 - \frac{2}{a} \right) \right\}. \tag{2.32}$$

This cancels the contribution from the bosonic fields. The system S is thus equivalent to $\tilde{S}$ after integrating out the auxiliary chiral multiplet, X, Φ and $\mathcal{F}$.

This argument for the equivalence between S and $\tilde{S}$, however, is valid in a formal sense, because the operators $D - \frac{2}{a}$ and thus $D_2 - \frac{2}{a}$ which appear in various places in the above expressions always have zero modes when the lattice volume is infinite [12, 30]. The kinetic operators of X and of $(\Phi, \mathcal{F})$ have zero eigenmodes *when tilded variables are kept fixed.* Thus the integration over the former gives zero and the latter gives infinity. On the other hand, kinetic operators in the effective action $\tilde{S}$ contain the factor $(1 - \frac{1}{2} a D_2)^{-1}$ which is a singular operator.

To see what is really happening here, it is instructive to consider the case of $m = g = 0$. In this case, integrations over Grassmann variables yield

$$\int \prod_x \mathrm{d}\chi(x) \mathrm{d}X(x)\, e^{-a^4 \sum_x \{ \frac{1}{2} \chi^T C D \chi - \frac{1}{a} X^T C X \}} = \mathrm{Pf}\{CD\}\, \mathrm{Pf} \left\{ -\frac{2}{a} C \right\}, \tag{2.33}$$

which is *not* singular in any sense. On the other hand, if we perform the integration over X first *while keeping* $\tilde{\chi} = \chi + X$, we have instead

$$\mathrm{Pf} \left\{ C \left(D - \frac{2}{a} \right) \right\} \int \prod_x \mathrm{d}\tilde{\chi}(x)\, e^{-a^4 \sum_x \{ \frac{1}{2} \tilde{\chi}^T C (1 - \frac{1}{2} a D_2)^{-1} D_1 \tilde{\chi} \}}, \tag{2.34}$$

which is of a structure of $0 \times \infty$, although nothing is wrong with the whole integral (2.33). The above argument simply shows that we are observing a non-singular object in an unnecessarily singular way. A similar argument is applied to the full action (2.20). Integrations over field variables do not produce any singularities. Only if we observe the integrations in a wrong way, seemingly singular natures as in eqs. (2.28) and (2.29) emerge. The full action S (2.20) and the action $\tilde{S}$ (2.28) are thus *inequivalent* by a singular quantity.[7]

[7]In this way, the no-go theorem of ref. [30] for a chiral invariant Yukawa interaction of the Majorana fermion is evaded by introducing auxiliary fields.

Nevertheless, we can infer that our formulation based on S and that based on $\tilde{S}$ are equivalent in *perturbation theory*. The point is that the free propagators among *tilded* variables

$$
\begin{aligned}
\langle\tilde{\chi}(x)\tilde{\chi}^T(y)\rangle C &= \frac{-D_1+(1-\frac{1}{2}aD_2)(m^*P_+ + mP_-)}{\frac{2}{a}D_2+(1-\frac{1}{2}aD_2)m^*m}a^{-4}\delta_{x,y}\,,\\
\langle\tilde{\phi}(x)\tilde{\phi}^*(y)\rangle &= \langle\tilde{\phi}^*(x)\tilde{\phi}(y)\rangle = \frac{-1}{\frac{2}{a}D_2+(1-\frac{1}{2}aD_2)m^*m}a^{-4}\delta_{x,y}\,,\\
\langle\tilde{\phi}(x)\tilde{F}(y)\rangle &= \langle\tilde{F}(x)\tilde{\phi}(y)\rangle = \frac{(1-\frac{1}{2}aD_2)m^*}{\frac{2}{a}D_2+(1-\frac{1}{2}aD_2)m^*m}a^{-4}\delta_{x,y}\,,\\
\langle\tilde{\phi}^*(x)\tilde{F}^*(y)\rangle &= \langle\tilde{F}^*(x)\tilde{\phi}^*(y)\rangle = \frac{(1-\frac{1}{2}aD_2)m}{\frac{2}{a}D_2+(1-\frac{1}{2}aD_2)m^*m}a^{-4}\delta_{x,y}\,,\\
\langle\tilde{F}(x)\tilde{F}^*(y)\rangle &= \langle\tilde{F}^*(x)\tilde{F}(y)\rangle = \frac{(1-\frac{1}{2}aD_2)\frac{2}{a}D_2}{\frac{2}{a}D_2+(1-\frac{1}{2}aD_2)m^*m}a^{-4}\delta_{x,y}\,,
\end{aligned}
\tag{2.35}
$$

which are directly obtained by using S, are identical to those obtained by using $\tilde{S}$ formally, i.e., by neglecting a singular nature of kinetic terms. Moreover, interaction vertices of S are identical to those of $\tilde{S}$ (in fact we have constructed S so that this is the case). Therefore, perturbative calculations based on S and that based on $\tilde{S}$ give rise to completely identical answers for correlation functions which have tilded variables in external lines. In a sense, our non-singular local lattice action S provides a natural justification for a prescription of refs. [11, 14] which utilizes the above form of propagators and interaction vertices of $\tilde{S}$. Of course, we think our formulation which includes the auxiliary chiral supermultiplet is superior at least formally, because it is manifestly free from singularities and it may have a meaning even as a non-perturbative formulation.

3. Supersymmetric Ward-Takahashi identity and its breaking

3.1 Derivation of a lattice Ward-Takahashi identity

We consider a structure of radiative corrections with the present lattice formulation of the Wess-Zumino model. As noted in the preceding section, in perturbation theory, our formulation is equivalent to the formulation of refs. [11, 12, 14]. One-loop radiative corrections in the latter formulation, in view of a realization of supersymmetry, had been extensively studied in ref. [11]. Here we study this issue in the continuum limit by using a Ward-Takahashi identity.

For a systematic study, it is quite helpful to introduce the one-particle irreducible (1PI) effective action Γ. Following the standard procedure, we introduce external sources for each elementary fields

$$
S_{\text{source}} = a^4\sum_x\{J_\chi\chi+J_\phi\phi+J_{\phi^*}\phi^*+J_FF+J_{F^*}F^*+J_XX+J_\Phi\Phi+J_{\Phi^*}\Phi^*+J_{\mathcal{F}}\mathcal{F}+J_{\mathcal{F}^*}\mathcal{F}^*\}\,.
\tag{3.1}
$$

We also introduce an external source K for a symmetry breaking of the action, $\delta_\epsilon S$, and a source L for a symmetry variation of $\delta_\epsilon S$, $\delta_{\epsilon'}\delta_\epsilon S$, and so on. Including these latter kind of external sources K, $L\dots$ only, we define the *total action*

$$S_{\mathrm{tot.}} = S - a^4 \sum \{K_\alpha (\tilde{\chi}^T C \Delta L)_\alpha + L_{\alpha,\beta} \delta_\alpha (\tilde{\chi}^T C \Delta L)_\beta + M_{\alpha\beta,\gamma} \delta_\alpha \delta_\beta (\tilde{\chi}^T C \Delta L)_\gamma + \cdots\}\,, \quad (3.2)$$

where δ_α stands for the symmetry variation with the transformation spinor parameter is removed:

$$\delta_\epsilon = \epsilon_\alpha \delta_\alpha\,. \quad (3.3)$$

The generating functional W of connected Green's functions is then defined by the functional integral

$$\begin{aligned} e^{-W} = & \int \prod_x \mathrm{d}\chi(x) \mathrm{d}\phi(x) \mathrm{d}\phi^*(x) \mathrm{d}F(x) \mathrm{d}F^*(x) \mathrm{d}X(x) \mathrm{d}\Phi(x) \mathrm{d}\Phi^*(x) \mathrm{d}\mathcal{F}(x) \mathrm{d}\mathcal{F}^*(x) \times \\ & \times e^{-S_{\mathrm{tot.}} - S_{\mathrm{source}}}\,. \end{aligned} \quad (3.4)$$

We then apply the Legendre transformation to W and change independent variables from external sources for elementary fields $(J_\chi, \dots)$ to the corresponding expectation values of elementary fields $(\langle\chi\rangle, \dots)$. In what follows, we denote expectation values by their original name as $\langle\chi\rangle \to \chi$ and so on for notational simplicity. We do *not* apply the Legendre transformation with respect to the sources $(K, L, M, \dots)$. In this way, we have the 1PI effective action

$$\Gamma = \Gamma[\chi, \phi, \phi^*, F, F^*, X, \Phi, \Phi^*, \mathcal{F}, \mathcal{F}^*; K, L, M, \dots], \quad (3.5)$$

which is a generating functional of 1PI Green's functions which include additional vertices coming from the second term of eq. (3.2).

Now, the action S is not invariant under the lattice super transformation (2.8) and (2.9), but it leaves the breaking (2.19). From this fact, we have the identity

$$\begin{aligned} \Big\langle & -a^4 \sum_x \{J_\chi \delta_\epsilon \chi + J_\phi \delta_\epsilon \phi + J_{\phi^*} \delta_\epsilon \phi^* + J_F \delta_\epsilon F + J_{F^*} \delta_\epsilon F^* + \\ & \qquad + J_X \delta_\epsilon X + J_\Phi \delta_\epsilon \Phi + J_{\Phi^*} \delta_\epsilon \Phi^* + J_{\mathcal{F}} \delta_\epsilon \mathcal{F} + J_{\mathcal{F}^*} \delta_\epsilon \mathcal{F}^*\} + \\ & + a^4 \sum_x \tilde{\chi}^T C \Delta L \epsilon + \\ & + a^4 \sum \left\{K_\alpha \epsilon_\beta \delta_\beta (\tilde{\chi}^T C \Delta L)_\alpha + L_{\alpha,\beta} \epsilon_\gamma \delta_\gamma \delta_\alpha (\tilde{\chi}^T C \Delta L)_\beta + \cdots\right\} \Big\rangle = 0\,. \end{aligned} \quad (3.6)$$

This identity, in terms of the 1PI effective action Γ, is expressed as

$$\begin{aligned} & -\sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial \chi} \{\sqrt{2} P_+ (D_1 \phi + F) \epsilon + \sqrt{2} P_- (D_1 \phi^* + F^*) \epsilon\} + \\ & \qquad + \sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial \phi} \sqrt{2} \epsilon^T C P_+ \chi + \sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial \phi^*} \sqrt{2} \epsilon^T C P_- \chi + \\ & + \sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial F} \sqrt{2} \epsilon^T C D_1 P_+ \chi + \sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial F^*} \sqrt{2} \epsilon^T C D_1 P_- \chi - \\ & - \sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial X} \{\sqrt{2} P_+ (D_1 \Phi + \mathcal{F}) \epsilon + \sqrt{2} P_- (D_1 \Phi^* + \mathcal{F}^*) \epsilon\} + \end{aligned}$$

$$
\begin{aligned}
&+\sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial \Phi}\sqrt{2}\epsilon^T C P_+ X + \sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial \Phi^*}\sqrt{2}\epsilon^T C P_- X + \\
&+\sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial \mathcal{F}}\sqrt{2}\epsilon^T C D_1 P_+ X + \sum_x \Gamma \frac{\overleftarrow{\partial}}{\partial \mathcal{F}^*}\sqrt{2}\epsilon^T C D_1 P_- X + \\
&+\sum_x \frac{\partial}{\partial K_\alpha}\Gamma \epsilon_\alpha + \\
&+\sum_x \left\{ K_\alpha \epsilon_\beta \frac{\partial}{\partial L_{\beta,\alpha}}\Gamma + L_{\alpha,\beta}\epsilon_\gamma \frac{\partial}{\partial M_{\gamma\alpha,\beta}}\Gamma + \cdots \right\} = 0 \,, \qquad (3.7)
\end{aligned}
$$

that is a linear equation of Γ. This is referred to as the *lattice* Ward-Takahashi identity. If the last two lines were not present in this expression, the above equation simply states that the effective action is invariant under the lattice analogue of super transformation, eqs. (2.8) and (2.9). Thus, contributions of these lines, especially contributions from the term,

$$
\sum_x \frac{\partial}{\partial K_\alpha}\Gamma \epsilon_\alpha \,, \qquad (3.8)
$$

that is the breaking of the supersymmetric Ward-Takahashi identity, will play a central role in our analysis below. Explicitly, this term is given by 1PI diagrams with insertions of the operator[8]

$$
\begin{aligned}
&-a^4 \sum_x \tilde{\chi}^T C \Delta L \epsilon = \\
&\quad = -a^4 \sum_x \tilde{\chi}^T C \sqrt{2}\left\{ g P_+ (2\tilde{\phi} D_1 \tilde{\phi} - D_1 \tilde{\phi}^2)\epsilon + g^* P_- (2\tilde{\phi}^* D_1 \tilde{\phi}^* - D_1 \tilde{\phi}^{*2})\epsilon \right\}. \qquad (3.9)
\end{aligned}
$$

The $a \to 0$ limit of these 1PI diagrams will be expressed by a *local* polynomial of field variables, because in this limit, the effect of our particular choice of regularization (the lattice artifact) should affect only local terms in the effective action Γ. Moreover, the operator (3.9) vanishes in the classical continuum limit (because the Leibniz rule holds in this limit) and it has no continuum analogue. Only when combined with ultraviolet divergences, $\langle \tilde{\chi}^T C \Delta L \epsilon \rangle_{\mathrm{1PI}}$ can acquire non-zero value. In these aspects, computation of $\langle \tilde{\chi}^T C \Delta L \epsilon \rangle_{\mathrm{1PI}}$ is similar to that of quantum anomalies, although this breaking of supersymmetry is not a genuine anomaly in a conventional sense.[9]

We expand Γ according to a number of internal loops of 1PI diagrams:

$$
\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_2 + \cdots \,. \qquad (3.10)
$$

The loop counting parameter in the present model is $g^* g$. The tree-level effective action Γ_0 is nothing but the total action (3.2),

$$
\Gamma_0 = S_{\mathrm{tot}}. \qquad (3.11)
$$

[8]Generally, Γ contains 1PI diagrams with multiple insertions of this operator. We will be interested in, however, terms of Γ that are linear in the external source K and consider 1PI diagrams with a single insertion of this operator below.

[9]It will be removed by local counter terms (supersymmetry is thought to be anomaly-free at least in perturbation theory) and also the structure of $\langle \tilde{\chi}^T C \Delta L \epsilon \rangle_{\mathrm{1PI}}$ is not universal, i.e., it will quite depend on a lattice formulation one adopts.

In fact, it is easy to see that $S_{\mathrm{tot.}}$ satisfies the lattice Ward-Takahashi identity. In this tree level approximation, the breaking term vanishes in the $a \to 0$ limit,

$$\lim_{a\to 0} \tilde{\chi}^T C \Delta L \epsilon = 0\,, \tag{3.12}$$

because the Leibniz rule holds in this limit. Thus the last two lines of the identity (3.7) vanish in the $a \to 0$ limit and the supersymmetry is restored in this classical continuum limit.

In loop diagrams, all external lines are tilded variables. The Ward-Takahashi identity for the effective action Γ_n $(n \geq 1)$ can thus be written as

$$\begin{aligned}
-\sum_x \Gamma_n \frac{\overleftarrow{\partial}}{\partial\tilde{\chi}}\{\sqrt{2}P_+(D_1\tilde{\phi}+\tilde{F})\epsilon+\sqrt{2}P_-(D_1\tilde{\phi}^*+\tilde{F}^*)\epsilon\}+&\\
+\sum_x \Gamma_n \frac{\overleftarrow{\partial}}{\partial\tilde{\phi}}\sqrt{2}\epsilon^T C P_+\tilde{\chi}+\sum_x \Gamma_n \frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^*}\sqrt{2}\epsilon^T C P_-\tilde{\chi}+&\\
+\sum_x \Gamma_n \frac{\overleftarrow{\partial}}{\partial\tilde{F}}\sqrt{2}\epsilon^T C D_1 P_+\tilde{\chi}+\sum_x \Gamma_n \frac{\overleftarrow{\partial}}{\partial\tilde{F}^*}\sqrt{2}\epsilon^T C D_1 P_-\tilde{\chi}+&\\
+\sum_x \frac{\partial}{\partial K_\alpha}\Gamma_n\epsilon_\alpha+&\\
+\sum_x \left\{K_\alpha\epsilon_\beta\frac{\partial}{\partial L_{\beta,\alpha}}\Gamma_n+L_{\alpha,\beta}\epsilon_\gamma\frac{\partial}{\partial M_{\gamma\alpha,\beta}}\Gamma_n+\cdots\right\}&=0\,.
\end{aligned} \tag{3.13}$$

We also recall that our system S possesses two global U(1) "symmetries": U(1) symmetry, eqs. (2.21) and (2.22), and $\mathrm{U}(1)_R$ symmetry, eqs. (2.24) and (2.25). In terms of the 1PI effective action Γ_n $(n \geq 1)$, these invariance can be expressed as

$$\sum_x\left\{\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}}\tilde{\phi}-\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^*}\tilde{\phi}^*-\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{F}}\tilde{F}+\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{F}^*}\tilde{F}^*\right\}-\Gamma_n\frac{\overleftarrow{\partial}}{\partial g}g+\Gamma_n\frac{\overleftarrow{\partial}}{\partial g^*}g^*=0\,, \tag{3.14}$$

and

$$\begin{aligned}
\sum_x\left\{\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\chi}}\gamma_5\tilde{\chi}-2\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}}\tilde{\phi}+2\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^*}\tilde{\phi}^*+4\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{F}}\tilde{F}-4\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{F}^*}\tilde{F}^*\right\}-&\\
-2\Gamma_n\frac{\overleftarrow{\partial}}{\partial m}m+2\Gamma_n\frac{\overleftarrow{\partial}}{\partial m^*}m^*=0\,,&
\end{aligned} \tag{3.15}$$

where we have set $K = L = \cdots = 0$. These identities are referred to as Ward-Takahashi identities associated to U(1) symmetries.

3.2 Supersymmetric Ward-Takahashi identity: improvement and renormalization

Supersymmetry is not exact in the present lattice formulation of the Wess-Zumino model. To achieve a supersymmetric continuum limit, we have to apply appropriate improvement and renormalization to the lattice action. In particular, in the continuum limit, a renormalized effective action must obey the *supersymmetric* Ward-Takahashi identity that is defined by

$$
\begin{aligned}
-\int \mathrm{d}^4x\, \Gamma_n \frac{\overleftarrow{\delta}}{\delta\tilde{\chi}}\{\sqrt{2}P_+(\gamma_\mu\partial_\mu\tilde{\phi}+\tilde{F})\epsilon+\sqrt{2}P_-(\gamma_\mu\partial_\mu\tilde{\phi}^*+\tilde{F}^*)\epsilon\}+ \\
+\int \mathrm{d}^4x\, \Gamma_n \frac{\overleftarrow{\partial}}{\partial\tilde{\phi}}\sqrt{2}\epsilon^T C P_+\tilde{\chi}+\int \mathrm{d}^4x\, \Gamma_n \frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^*}\sqrt{2}\epsilon^T C P_-\tilde{\chi}+ \\
+\int \mathrm{d}^4x\, \Gamma_n \frac{\overleftarrow{\partial}}{\partial\tilde{F}}\sqrt{2}\epsilon^T C\gamma_\mu\partial_\mu P_+\tilde{\chi}+\int \mathrm{d}^4x\, \Gamma_n \frac{\overleftarrow{\partial}}{\partial\tilde{F}^*}\sqrt{2}\epsilon^T C\gamma_\mu\partial_\mu P_-\tilde{\chi}=0\,,
\end{aligned}
\tag{3.16}
$$

and[10]

$$
\int \mathrm{d}^4x\, \frac{\delta}{\delta K_\alpha}\Gamma_n=\int \mathrm{d}^4x\, \frac{\delta}{\delta L_{\beta,\alpha}}\Gamma_n=\int \mathrm{d}^4x\, \frac{\delta}{\delta M_{\gamma\alpha,\beta}}\Gamma_n=\cdots=0\,. \tag{3.17}
$$

Also the U(1) symmetries must be preserved

$$
\int \mathrm{d}^4x\, \left\{\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}}\tilde{\phi}-\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^*}\tilde{\phi}^*-\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{F}}\tilde{F}+\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{F}^*}\tilde{F}^*\right\}-\Gamma_n\frac{\overleftarrow{\partial}}{\partial g}g+\Gamma_n\frac{\overleftarrow{\partial}}{\partial g^*}g^*=0\,, \tag{3.18}
$$

and

$$
\begin{aligned}
\int \mathrm{d}^4x\, \left\{\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\chi}}\gamma_5\tilde{\chi}-2\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}}\tilde{\phi}+2\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{\phi}^*}\tilde{\phi}^*+4\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{F}}\tilde{F}-4\Gamma_n\frac{\overleftarrow{\partial}}{\partial\tilde{F}^*}\tilde{F}^*\right\}- \\
-2\Gamma_n\frac{\overleftarrow{\partial}}{\partial m}m+2\Gamma_n\frac{\overleftarrow{\partial}}{\partial m^*}m^*=0\,.
\end{aligned}
\tag{3.19}
$$

In these expressions, all fields, external sources, mass parameters and coupling constants are regarded as *renormalized* quantities. To achieve this supersymmetric finite continuum theory, we take following steps. (1) We compute 1PI nth loop diagrams Γ_n by using the total action $S_{\mathrm{tot.}}$. (2) We add appropriate local counter terms to the total action $S_{\mathrm{tot.}}$ so that the supersymmetric Ward-Takahashi identities (3.16) and (3.17) hold in the $a\to 0$ limit. At this stage, all fields and parameters in eqs. (3.16) and (3.17) are understood as bare quantities. This improvement step removes supersymmetry breaking due to lattice artifacts in our formulation. (3) Then we further add appropriate local counter terms to the total action $S_{\mathrm{tot.}}$ so that Γ_n is finite in the $a\to 0$ limit. This step corresponds to the standard supersymmetric renormalization and we assume its validity, i.e., we assume a renormalizability of this lattice model. Explicitly, we modify the total action as

$$
\begin{aligned}
S_{\mathrm{tot.}}\to S_{\mathrm{tot.}}+a^4\sum_x Z_n\bigg\{\frac{1}{2}\chi^T CD\chi+\phi^* D_1^2\phi+F^*F+ \\
+FD_2\phi+F^*D_2\phi^*-\frac{1}{a}X^T CX-\frac{2}{a}(\mathcal{F}\Phi+\mathcal{F}^*\Phi^*)\bigg\}\,,
\end{aligned}
\tag{3.20}
$$

[10]This requirement may seem somewhat ad hoc. However, without this kind of additional condition, divergences arise from sub-diagrams in a 1PI diagram which contains source terms K, L, M, $\ldots$ cannot be cancelled by an addition of local terms to Γ_{n+1}. Note that this requirement is satisfied in the tree level.

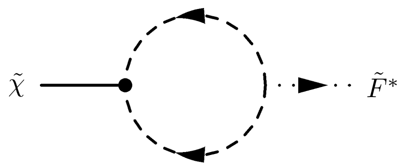

Figure 1: A diagram which contributes to eq. (3.21). The blob indicates the supersymmetry breaking operator (3.9). The broken line is the propagator of $\tilde{\phi}$ and the arrow indicates a flow of the chirality.

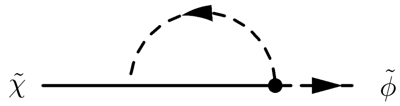

Figure 2: A diagram which possibly contributes to eq. (3.21). The bold line is the propagator of $\tilde{\chi}$. The $a \to 0$ limit of this diagram turns out to be vanishing.

where Z_n is a common wave function renormalization factor. (4) All these steps must be consistent with global U(1) symmetries. We then repeat the above steps for 1PI diagrams of one loop higher, Γ_{n+1}, by using $S_{\text{tot.}}$ so determined.

The tree-level effective action Γ_0 is given by the total action in eq. (3.2) and of course it does not need any renormalization and improvement. To see the situation in the one-loop level, we evaluate in the next subsection the supersymmetry breaking term in the lattice Ward-Takahashi identity (3.13) to this order.

3.3 One-loop evaluation of the breaking term

We evaluate the supersymmetry breaking term in eq. (3.13) in the one-loop order. We set $K = L = \cdots = 0$. It is given by one-loop 1PI diagrams with a single insertion of the operator (3.9). A computation of the continuum limit of these one-loop 1PI diagrams is not difficult, if one invokes a powerful Reisz's theorem [31, 32] on lattice Feynman integrals.

Most singular one-loop diagrams which possibly contribute to

$$\lim_{a\to 0}\sum_x \frac{\partial}{\partial K_\alpha}\Gamma_1\epsilon_\alpha\Big|_{K=L=\cdots=0} = \lim_{a\to 0}\left\{-a^4\sum_x\langle\tilde{\chi}^T C\Delta L\rangle_{\text{1PI}}^{\text{1loop}}\epsilon\Big|_{K=L=\cdots=0}\right\}, \tag{3.21}$$

are given by figures 1–4 and their complex conjugate.

By using propagators in eq. (2.35), the contribution from figure 1 to eq. (3.21) is given by

$$+2\sqrt{2}g^* g a^4\sum_x a^4\sum_y \tilde{\chi}^T(x)CP_+\tilde{F}^*(y)\int_{\mathcal{B}}\frac{\mathrm{d}^4 q}{(2\pi)^4}\,e^{iq(x-y)}\int_{\mathcal{B}}\frac{\mathrm{d}^4 k}{(2\pi)^4}\,\mathcal{I}_1(k,q;m,a)\epsilon\,, \tag{3.22}$$

where $\mathcal{B}$ denotes the Brillouin zone

$$\mathcal{B} = \left\{p\in R^4 \mid |p_\mu| \le \frac{\pi}{a}\right\}, \tag{3.23}$$

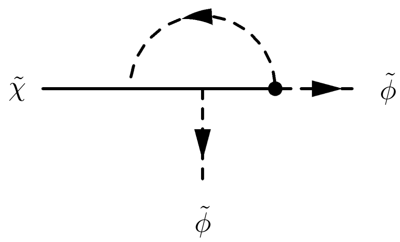

Figure 3: A diagram which possibly contributes to eq. (3.21). The $a \to 0$ limit of this diagram turns out to be vanishing due to the $\mathrm{U}(1)_R$ symmetry of the $m = 0$ case.

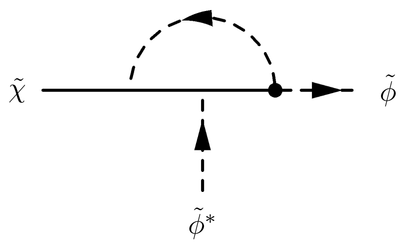

Figure 4: A diagram which possibly contributes to eq. (3.21). The $a \to 0$ limit of this diagram turns out to be vanishing due to the $\mathrm{U}(1)_R$ symmetry of the $m = 0$ case.

and the integrand is given by

$$\begin{aligned}
&\mathcal{I}_1(k, q; m, a) = \\
&\quad = \frac{1}{\frac{2}{a}\widetilde{D}_2(k) + \left(1 - \frac{1}{2}a\widetilde{D}_2(k)\right) m^* m} \frac{\widetilde{D}_1(k+q) - \widetilde{D}_1(k) - \widetilde{D}_1(q)}{\frac{2}{a}\widetilde{D}_2(k+q) + \left(1 - \frac{1}{2}a\widetilde{D}_2(k+q)\right) m^* m} . \qquad (3.24)
\end{aligned}$$

In this expression, $\widetilde{D}_1$ and $\widetilde{D}_2$ denote the momentum representation of difference operators in eq. (2.10)

$$D_1 e^{ipx} = \widetilde{D}_1(p) e^{ipx} , \qquad D_2 e^{ipx} = \widetilde{D}_2(p) e^{ipx} , \qquad (3.25)$$

and the explicit forms are given by

$$\begin{aligned}
\widetilde{D}_1(p) &= i \sum_\mu \gamma_\mu \mathring{p}_\mu \left\{ 1 + \frac{1}{2} a^4 \sum_{\nu<\rho} \hat{p}_\nu^2 \hat{p}_\rho^2 \right\}^{-1/2} , \\
\widetilde{D}_2(p) &= \frac{1}{a} \left\{ 1 - \left(1 - \frac{1}{2} a^2 \sum_\mu \hat{p}_\mu^2 \right) \left\{ 1 + \frac{1}{2} a^4 \sum_{\nu<\rho} \hat{p}_\nu^2 \hat{p}_\rho^2 \right\}^{-1/2} \right\} , \qquad (3.26)
\end{aligned}$$

with abbreviations

$$\mathring{p}_\mu = \frac{1}{a} \sin(a p_\mu) , \qquad \hat{p}_\mu = \frac{2}{a} \sin\left(\frac{a p_\mu}{2} \right) . \qquad (3.27)$$

We note that both $\mathring{p}_\mu$ and $\hat{p}_\mu$ reduce to the momentum in the continuum theory in the $a \to 0$ limit, $\lim_{a\to 0} \mathring{p}_\mu = \lim_{a \to 0} \hat{p}_\mu = p_\mu$.

Now, a crucial idea of the Reisz power counting theorem is to consider the $\lambda \to \infty$ limit of the integrand, after replacing the internal loop momenta k_i by λk_i and the lattice spacing a by a/λ. From the above explicit form of the integrand, we find

$$\mathcal{I}_1\left(\lambda k, q; m, \frac{a}{\lambda}\right) = O\left(\frac{1}{\lambda^4}\right), \tag{3.28}$$

in the $\lambda \to \infty$ limit. This implies that the degree of divergence of the above loop integral (in a sense of the Reisz power counting theorem) is 0 and the $a \to 0$ limit of the loop integral may exhibit a logarithmic divergence of the form $\log a$.

To reduce the degree of divergence, we thus take the first term in the Taylor expansion of the integrand with respect to the external momentum q and consider a subtraction of the form

$$\mathcal{I}_1(k, q; m, a) - \mathcal{I}_1(k, 0; m, a)\,. \tag{3.29}$$

However, since $\mathcal{I}_1(k, 0; m, a) = 0$, this subtraction does not improve the convergence behavior at all.

We are thus lead to consider a subtraction to the next order term in the Taylor expansion

$$\mathcal{I}_1(k, q; m, a) - \mathcal{I}_1(k, 0; m, a) - q_\mu \partial^q_\mu \mathcal{I}_1(k, 0; m, a)\,. \tag{3.30}$$

Then we find that this combination behaves as $O(1/\lambda^6)$ in the $\lambda \to \infty$ limit defined above. The Reisz power counting theorem then states that the $a \to 0$ of the loop integral of this combination is convergent and moreover the limit is given by[11]

$$\begin{aligned}
&\lim_{a\to 0} \int_{\mathcal{B}} \frac{\mathrm{d}^4 k}{(2\pi)^4} \left\{\mathcal{I}_1(k, q; m, a) - \mathcal{I}_1(k, 0; m, a) - q_\mu \partial^q_\mu \mathcal{I}_1(k, 0; m, a)\right\} = \\
&\quad = \int_{R^4} \frac{\mathrm{d}^4 k}{(2\pi)^4} \lim_{a\to 0} \left\{\mathcal{I}_1(k, q; m, a) - \mathcal{I}_1(k, 0; m, a) - q_\mu \partial^q_\mu \mathcal{I}_1(k, 0; m, a)\right\} \\
&\quad = 0\,.
\end{aligned} \tag{3.31}$$

The last equality follows from a property of the operator ΔL (3.9) that it vanishes in the classical continuum limit.

In this way, we obtain

$$\begin{aligned}
&\lim_{a\to 0} \int_{\mathcal{B}} \frac{\mathrm{d}^4 k}{(2\pi)^4} \mathcal{I}_1(k, q; m, a) = \\
&\quad = \lim_{a\to 0} \int_{\mathcal{B}} \frac{\mathrm{d}^4 k}{(2\pi)^4} \left\{\mathcal{I}_1(k, 0; m, a) + q_\mu \partial^q_\mu \mathcal{I}_1(k, 0; m, a)\right\} + \\
&\qquad + \lim_{a\to 0} \int_{\mathcal{B}} \frac{\mathrm{d}^4 k}{(2\pi)^4} \left\{\mathcal{I}_1(k, q; m, a) - \mathcal{I}_1(k, 0; m, a) - q_\mu \partial^q_\mu \mathcal{I}_1(k, 0; m, a)\right\} \\
&\quad = \int_{-\pi}^{\pi} \frac{\mathrm{d}^4 k}{(2\pi)^4} \lim_{a\to 0} \frac{1}{a^4} q_\mu \partial^q_\mu \mathcal{I}_1(k/a, 0; m, a) \\
&\quad = -\frac{r_1}{2} i\gamma_\mu q_\mu\,,
\end{aligned} \tag{3.32}$$

[11] In ref. [3], this fact is referred to as an empirical "rule".

where the coefficient r_1 is given by an integral

$$r_1 = \frac{-1}{32\pi^4}\int_{-\pi}^{\pi} \mathrm{d}^4k \left\{\left(1-\frac{1}{2}\hat{k}_0^2\right)B^{-1/2} - \frac{1}{2}\mathring{k}_0^2\sum_{\mu\neq 0}\hat{k}_\mu^2\,B^{-3/2} - 1\right\}\times$$
$$\times\left\{1-\left(1-\frac{1}{2}\sum_{\mu}\hat{k}_\mu^2\right)B^{-1/2}\right\}^{-2}, \qquad (3.33)$$

with abbreviations

$$\mathring{k}_\mu = \sin k_\mu, \qquad \hat{k}_\mu = 2\sin\left(\frac{k_\mu}{2}\right), \qquad B = 1+\frac{1}{2}\sum_{\mu<\nu}\hat{k}_\mu^2\hat{k}_\nu^2\,. \qquad (3.34)$$

From a numerical integration, we have

$$r_1 = +0.1518\,. \qquad (3.35)$$

Eq. (3.32) then implies that figure 1 gives rise to

$$\lim_{a\to 0}\left\{-a^4\sum_x\langle\tilde{\chi}^T C\Delta L\rangle_{\mathrm{1PI}}^{\mathrm{1loop}}\epsilon\Big|_{K=L=\cdots=0}\right\} =$$
$$= -r_1 g^* g\int \mathrm{d}^4x\,\left\{\tilde{F}^*\sqrt{2}\epsilon^T C\gamma_\mu\partial_\mu P_+\tilde{\chi} + \tilde{F}\sqrt{2}\epsilon^T C\gamma_\mu\partial_\mu P_-\tilde{\chi}\right\}. \qquad (3.36)$$

Next, we consider the contribution of figure 2 which is given by

$$+4\sqrt{2}g^* g a^4\sum_x a^4\sum_y\tilde{\chi}^T(x)CP_-\tilde{\phi}(y)\int_{\mathcal{B}}\frac{\mathrm{d}^4q}{(2\pi)^4}\,e^{iq(x-y)}\int_{\mathcal{B}}\frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathcal{I}_2(k,q;m,a)\epsilon\,, \qquad (3.37)$$

where

$$\mathcal{I}_2(k,q;m,a) = \qquad (3.38)$$
$$= \frac{\widetilde{D}_1(k)}{\frac{2}{a}\widetilde{D}_2(k)+\left(1-\frac{1}{2}a\widetilde{D}_2(k)\right)m^*m}\,\frac{\widetilde{D}_1(k+q)-\widetilde{D}_1(k)-\widetilde{D}_1(q)}{\frac{2}{a}\widetilde{D}_2(k+q)+\left(1-\frac{1}{2}a\widetilde{D}_2(k+q)\right)m^*m}\,.$$

We find that $\mathcal{I}_2(\lambda k,q;m,a/\lambda) = O(1/\lambda^3)$ in the $\lambda\to\infty$ limit and the degree of divergence of the above integral is 1. A twice subtraction

$$\mathcal{I}_2(k,q;m,a) - \mathcal{I}_2(k,0;m,a) - q_\mu\partial_\mu^q\mathcal{I}_2(k,0;m,a)\,, \qquad (3.39)$$

then makes the degree of divergence -1 and the $a\to 0$ limit of the loop integral convergent. Again, as before, the $a\to 0$ limit of the combination (3.39) vanishes. However, this time, $\mathcal{I}_2(k,0;m,a) = 0$ and the remaining lattice integral vanishes too:

$$\int_{\mathcal{B}}\frac{\mathrm{d}^4k}{(2\pi)^4}\,q_\mu\partial_\mu^q\mathcal{I}_2(k,0;m,a) = 0\,. \qquad (3.40)$$

Thus figure 2 has no contribution in the $a\to 0$ limit.

The contribution of figure 3 is given by

$$-8\sqrt{2}m^*g^*g^2a^4\sum_x a^4\sum_y a^4\sum_z \tilde{\chi}^T(x)CP_-\tilde{\phi}(y)\tilde{\phi}(z)\int_{\mathcal{B}}\frac{\mathrm{d}^4q_1}{(2\pi)^4}e^{iq_1(x-z)}\int_{\mathcal{B}}\frac{\mathrm{d}^4q_2}{(2\pi)^4}e^{iq_2(x-y)}\times$$
$$\times\int_{\mathcal{B}}\frac{\mathrm{d}^4k}{(2\pi)^4}\mathcal{I}_3(k,q_1,q_2;m,a)\epsilon\,, \tag{3.41}$$

where

$$\mathcal{I}_3(k,q_1,q_2;m,a)=$$
$$=\frac{\widetilde{D}_1(k-q_2)}{\frac{2}{a}\widetilde{D}_2(k-q_2)+\left(1-\frac{1}{2}a\widetilde{D}_2(k-q_2)\right)m^*m}\frac{1-\frac{a}{2}\widetilde{D}_2(k)}{\frac{2}{a}\widetilde{D}_2(k)+\left(1-\frac{1}{2}a\widetilde{D}_2(k)\right)m^*m}\times$$
$$\times\frac{\widetilde{D}_1(k+q_1)-\widetilde{D}_1(k)-\widetilde{D}_1(q_1)}{\frac{2}{a}\widetilde{D}_2(k+q_1)+\left(1-\frac{1}{2}a\widetilde{D}_2(k+q_1)\right)m^*m}\,. \tag{3.42}$$

This behaves as $O(1/\lambda^5)$ in the $\lambda\to\infty$ limit defined above (the degree of divergence is -1) and the $a\to 0$ limit of the loop integral converges without any subtraction. Since $\lim_{a\to 0}\mathcal{I}_3(k,q;m,a)=0$, however, the contribution of figure 3 vanishes in the $a\to 0$ limit. An underlying reason for this good convergence behavior of figure 3 is the chiral $\mathrm{U}(1)_R$ symmetry of S with $m=0$ under eq. (2.23). Due to this symmetry, this diagram vanishes for $m=0$ even with $a\neq 0$. In fact, if this diagram had a contribution to the breaking term when $m=0$, through the Ward-Takahashi identity (3.13), there must be terms of the form $\tilde{\phi}^*\tilde{\phi}^2$ or $\tilde{F}^*\tilde{\phi}^2$ in the one-loop effective action Γ_1: Both are however forbidden by the chiral $\mathrm{U}(1)_R$ symmetry (2.24). Similarly we find that figure 4 has no contribution in the $a\to 0$ limit.

As is also clear from above expressions, the vertex for the breaking term $\tilde{\chi}^TC\Delta L\epsilon$ in eq. (3.9) in the momentum space (see figure 5) is proportional to a combination $\widetilde{D}_1(p_1)+\widetilde{D}_1(p_2)+\widetilde{D}_1(p_3)$. From the fact that in one-loop diagrams one of three legs of the vertex is always an external line and from the momentum conservation at the vertex, we find that this vertex, when inserted in a one-loop diagram, gives rise to $O(\lambda^0)$ factor in the $\lambda\to\infty$ limit defined above. This implies that this vertex effectively behaves as a non-derivative coupling in the power counting argument for one-loop diagrams. Then it is easy to confirm that the degree of divergence of a loop integral in all possible one-loop diagrams other than those in figures 1–4 is negative. By repeating a similar argument as above, we then infer that the $a\to 0$ limit of those contributions is zero.

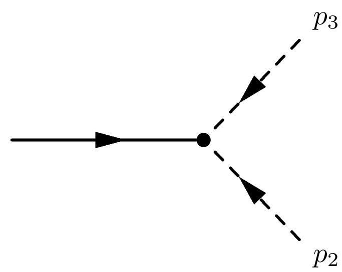

Figure 5: The supersymmetry breaking vertex (3.9) in the momentum space.

In summary, only the contribution of figure 1 survives in the $a\to 0$ limit and we have

$$\lim_{a\to 0}\sum_x\frac{\partial}{\partial K_\alpha}\Gamma_1\epsilon_\alpha\Big|_{K=L=\cdots=0}=$$
$$=-r_1g^*g\int\mathrm{d}^4x\,\left\{\tilde{F}^*\sqrt{2}\epsilon^TC\gamma_\mu\partial_\mu P_+\tilde{\chi}+\tilde{F}\sqrt{2}\epsilon^TC\gamma_\mu\partial_\mu P_-\tilde{\chi}\right\}. \tag{3.43}$$

As a general argument shows, the supersymmetry breaking in the $a \to 0$ limit is a local polynomial of fields.

3.4 One-loop level improvement and the renormalization

From the above one-loop calculation (3.43), we can extract following information. First, combining it with the $a \to 0$ limit of the lattice Ward-Takahashi identity (3.13), we find that the coefficient of the term $\int \mathrm{d}^4x\, \tilde{F}^*\tilde{F}$ in Γ_1 is different from the supersymmetric value. Namely, the coefficient of this term is not consistent with the supersymmetric Ward-Takahashi identity (3.16). Next, eq. (3.43) shows that Γ_1 contains a finite term of the form $\int \mathrm{d}^4x\, \{\tilde{F}^*\sqrt{2}K^T C\gamma_\mu\partial_\mu P_+\tilde{\chi} + \text{c.c.}\}$ which is not consistent with eq. (3.17). These two are *only* places in Γ_1 in which the breaking of supersymmetry appears in the continuum limit (to the order $O(K, L^0, \ldots)$). According to the general strategy of section 3.2, we thus modify the total action $S_{\text{tot.}}$ to

$$S_{\text{tot.}} - a^4 \sum_x r_1 g^* g \left\{\tilde{F}^*\tilde{F} + \tilde{F}^*\sqrt{2}K^T C D_1 P_+\tilde{\chi} + \tilde{F}\sqrt{2}K^T C D_1 P_-\tilde{\chi}\right\} + \\ + O(K^2, L, \ldots) \tag{3.44}$$

to restore the supersymmetric Ward-Takahashi identity (3.16). The added terms contribute to Γ_1.

In ref. [11], one-loop 1PI two point functions are computed and it was found that radiative corrections to kinetic terms of $\tilde{\chi}$, $\tilde{\phi}$ and $\tilde{F}$ are in general different by finite amount, although logarithmically diverging parts are of an identical magnitude. Our observation above is consistent with this result and is slightly stronger: We observed that only the wave function renormalization of the auxiliary field $\tilde{F}$ differs from other two in the continuum limit. The improvement above adjusts this discrepancy in wave function renormalization factors.

After the above improvement, the effective action Γ_1 becomes supersymmetric. The standard statements concerning the supersymmetric Wess-Zumino model are then applied to the one-loop effective action Γ_1. For example, wave function renormalization factors for $\tilde{\chi}$, $\tilde{\phi}$ and $\tilde{F}$ are common as indicated in eq. (3.20). Also, a local term of the form

$$\int \mathrm{d}^4x\, V(\tilde{\phi}^*, \tilde{\phi})\,, \tag{3.45}$$

where V is an arbitrary local polynomial of $\tilde{\phi}$ and $\tilde{\phi}^*$ without any derivatives, does not appear in the one-loop effective action Γ_1, simply because such a combination is not supersymmetric, i.e, it is not a solution to the supersymmetric Ward-Takahashi identity (3.16).[12] This conclusion is again consistent with a one-loop analysis of ref. [11] that no terms consisting only of ϕ and ϕ^* are generated by one-loop radiative corrections.[13]

[12] An exception in this argument is $V =$ const. that is nothing but the cosmological term. One cannot exclude the cosmological term from the supersymmetric Ward-Takahashi identity alone. In the present model, the cosmological term vanishes in the one-loop level (see the appendix), as expected from an *exact* supersymmetry of the free action S_0.

[13] The result of ref. [11] is somewhat stronger: Up to the quartic order in $\tilde{\phi}$ or $\tilde{\phi}^*$, it was observed that such terms are not generated even for $a \neq 0$.

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By a similar reasoning, we can also show the non-renormalization theorem [34]–[37] of the form in ref. [37]. The non-renormalization theorem states that the F term of the structure

$$\int \mathrm{d}^4x \left\{ \frac{1}{2}\tilde{\chi}^T C W''(\tilde{\phi}) P_+ \tilde{\chi} + \tilde{F} W'(\tilde{\phi}) \right\}, \tag{3.46}$$

where $W(\tilde{\phi})$ is the superpotential, is not generated by radiative corrections, although this *is* a supersymmetric combination. We first note that, from a structure of one-loop diagrams, dependences of such a term on the coupling constant must be of the form

$$\int \mathrm{d}^4x \, \{ g\tilde{F} W'(g\tilde{\phi}) + \cdots \} . \tag{3.47}$$

The complex conjugate g^* cannot appear here from a structure of interaction vertices. Next we recall that our lattice action S possesses an exact U(1) symmetry under eqs. (2.21) and (2.22). The above allowed structure (3.47) is, however, inconsistent with this U(1) symmetry.[14] Thus we conclude that the F term (3.46) as the whole cannot be generated by one-loop radiative corrections. This conclusion is again consistent with the analysis of ref. [11]; there it was observed that terms such as $\tilde{\chi}^T C \tilde{\chi}$ and $\tilde{\chi}^T C \tilde{\phi} P_+ \tilde{\chi}$ are not generated in the one-loop order.

Finally, by further adding local counter terms (3.20), the one-loop effective action $\mathit{\Gamma}_1$ is made finite, i.e., a supersymmetric renormalized theory is defined. Obviously the theory so defined preserves the U(1) symmetries, i.e., eqs. (3.18) and (3.19) hold, because all stages of the above procedure (and the lattice regularization itself) do not affect these symmetries.

3.5 Higher loops

In the one-loop order, we have observed that if we add local terms of the form (3.44) to the total action, especially by an adjustment of the term $\tilde{F}^*\tilde{F}$, supersymmetry is restored in the continuum limit. In this subsection, we consider if this simple situation persists to higher orders of perturbation theory. We will find that, instead of a single combination of local terms (3.44), there are at most 9 combinations that we have to take into account for an improvement of higher loop effective action $\mathit{\Gamma}_n$. The improvement in higher loops is thus much more involved. We note, however, this number of combinations which require adjustments is much less than that in the formulation based on the Wilson fermion [3] due to the exact $\mathrm{U}(1)_R$ symmetry (2.24) and (2.25) in the present formulation.

Suppose that the procedure in section 3.2 of the renormalization and the improvement (which may require an addition of 9 combinations of local terms to $S_{\mathrm{tot.}}$) work for $\mathit{\Gamma}_1$, $\ldots$, $\mathit{\Gamma}_{n-1}$. Now take a 1PI n loop diagram γ_n which contains a single insertion of the operator (3.9). From the above assumption, all 1PI *sub-diagrams* are already made finite by the renormalization of $\mathit{\Gamma}_1$, $\ldots$, $\mathit{\Gamma}_{n-1}$. By applying a standard power counting argument to the present case, the superficial degree of divergence of such a diagram γ_n is given by

$$\omega(\gamma_n) = \frac{9}{2} - \frac{3}{2}E_\chi - E_\phi - 2E_F - I_{\phi F} , \tag{3.48}$$

[14] Our argument here is somewhat similar to that of ref. [38]. The conclusion here, however, is just a reflection of a simple fact that there is *no* 1PI one-loop diagram made out from only $\tilde{F}$ and $\tilde{\phi}$ external lines.

where E_χ, E_ϕ, E_F denote a number of external lines of $\tilde{\chi}$, $\tilde{\phi}$ or $\tilde{\phi}^*$, $\tilde{F}$ or $\tilde{F}^*$, respectively and $I_{\phi F}$ denotes a number of $\tilde{\phi}\tilde{F}$ and $\tilde{\phi}^*\tilde{F}^*$ type internal lines. To derive this formula, one has to note that the propagator $\langle \tilde{F}\tilde{F}^* \rangle$ behaves as $O(\lambda^0)$ (no suppression factor) in the Reisz power counting rule.

Also, in deriving the above formula, we have noted a fact that an insertion of the vertex (3.9) in higher loop diagrams effectively behaves as $O(\lambda)$ in the Reisz power counting rule, because in higher loop diagrams all momenta in figure 5 can simultaneously become large. In one-loop diagrams, on the other hand, the vertex figure 5 behaves as $O(\lambda^0)$ because one of three legs must always be an external line and momentum from the external line is kept fixed in the $\lambda \to \infty$ limit. (For one-loop diagrams, 9/2 in the formula (3.48) is changed to 7/2; see below.) This difference in a behavior of the supersymmetry breaking term in the $\lambda \to \infty$ limit in one-loop and higher-loop diagrams is crucial and due to this difference much more combinations have to be included in local counter terms to obtain a supersymmetric continuum limit.

Now, since we have assumed that all 1PI sub-diagrams of γ_n are made finite by a renormalization, the above superficial degree of divergence will be an overall degree of divergence. Then if $\omega(\gamma_n) < 0$, the Reisz theorem states that the $a \to 0$ limit of the diagram γ_n is given by R^4 integrations of the $a \to 0$ limit of the integrand, as we have seen in eq. (3.31). However, due to the vertex (3.9), the $a \to 0$ limit of the integrand always vanishes. Hence diagrams which can contribute to the supersymmetry breaking in the continuum limit must possess $\omega(\gamma_n) \geq 0$. Noting for γ_n, $E_\chi = 1, 3, \dots$, we can see that there are seven combinations for $\omega(\gamma_n) \geq 0$, i.e., $(E_\chi, E_\phi, E_F) = (1,0,0)$, $(1,1,0)$, $(1,2,0)$, $(1,3,0)$, $(1,0,1)$, $(1,1,1)$ and $(3,0,0)$. The total mass dimension of γ_n is 9/2.

A structure of γ_n moreover must be consistent with exact global symmetries in the present formulation; the $\mathrm{U}(1)_R$ symmetry, eqs. (2.24) and (2.25), and the $\mathrm{U}(1)$ symmetry, eqs. (2.21) and (2.22). The $\mathrm{U}(1)_R$ and the $\mathrm{U}(1)$ charges of γ_n are $(-3,+1)$ or $(+3,-1)$.

Finally, we have to check a consistency of γ_n with the lattice Ward-Takahashi identity (3.13). By examining various possible local terms in Γ_n which also must be consistent with the global symmetries, we finally find that a most general form of

$$\lim_{a\to 0} \sum_x \frac{\partial}{\partial K_\alpha} \Gamma_n \epsilon_\alpha \Big|_{K=L=\cdots=0}, \tag{3.49}$$

is given by[15]

$$\begin{aligned} -(g^*g)^n \int d^4x \Big\{ & r_n \tilde{F}^* \sqrt{2}\epsilon^T C\gamma_\mu \partial_\mu P_+ \tilde{\chi} + \\ & + \Big(s_n \partial_\mu^2 \tilde{\phi}^* + \frac{1}{a^2} t_n \tilde{\phi}^* + 2u_n \tilde{\phi}^{*2}\tilde{\phi} + \\ & \quad + \frac{1}{a^2} v_n \frac{m^*}{g^*} + w_n \left[2\frac{m^*}{g^*}\tilde{\phi}^*\tilde{\phi} + \frac{m}{g}\tilde{\phi}^{*2} \right] + 2x_n \frac{m^{*2}}{g^{*2}}\tilde{\phi} \Big) \sqrt{2}\epsilon^T C P_+ \tilde{\chi} + \end{aligned}$$

[15] To list up possible combinations, it is easier to work out the massless case $m = 0$ first and then restore possible dependences on m by substituting $\tilde{\phi} \to m/g$ and $\tilde{\phi}^* \to m^*/g^*$ in arbitrary ways.

$$
\begin{aligned}
&+ y_n(2g\tilde{F}\tilde{\phi}\sqrt{2}\epsilon^T CP_+\tilde{\chi} + g\tilde{\phi}^2\sqrt{2}\epsilon^T C\gamma_\mu\partial_\mu P_+\tilde{\chi}) + \\
&+ z_n(m\tilde{F}\sqrt{2}\epsilon^T CP_+\tilde{\chi} + m\tilde{\phi}\sqrt{2}\epsilon^T C\gamma_\mu\partial_\mu P_+\tilde{\chi})\Big\}
\end{aligned} \tag{3.50}
$$

plus its complex conjugate (the projection operator is replaced by $P_+ \to P_-$). The real coefficients $r_n, \ldots, z_n$ are given by dimensionless polynomials of $\log(a^2m^*m)$ and a^2m^*m.

Through the $a \to 0$ limit of the lattice Ward-Takahashi identity, the above form of the breaking term implies that the supersymmetry breaking terms in the effective action take the form:

$$
\begin{aligned}
&+(g^*g)^n \int d^4x \Big\{ r_n\tilde{F}^*\tilde{F} + s_n\tilde{\phi}^*\partial_\mu^2\tilde{\phi} + \frac{1}{a^2}t_n\tilde{\phi}^*\tilde{\phi} + u_n\tilde{\phi}^{*2}\tilde{\phi}^2 + \frac{1}{a^2}v_n\left(\frac{m^*}{g^*}\tilde{\phi} + \frac{m}{g}\tilde{\phi}^*\right) + \\
&\qquad + w_n\left(\frac{m^*}{g^*}\tilde{\phi}^*\tilde{\phi}^2 + \frac{m}{g}\tilde{\phi}^{*2}\tilde{\phi}\right) + x_n\left(\frac{m^{*2}}{g^{*2}}\tilde{\phi}^2 + \frac{m^2}{g^2}\tilde{\phi}^{*2}\right) + \\
&\qquad + y_n(g\tilde{F}\tilde{\phi}^2 + g^*\tilde{F}^*\tilde{\phi}^{*2}) + z_n(m\tilde{F}\tilde{\phi} + m^*\tilde{F}^*\tilde{\phi}^*)\Big\} + \\
&+ (g^*g)^n \int d^4x \Big\{ r_n\tilde{F}^*\sqrt{2}K^TC\gamma_\mu\partial_\mu P_+\tilde{\chi} + \\
&\qquad + \Big(s_n\partial_\mu^2\tilde{\phi}^* + \frac{1}{a^2}t_n\tilde{\phi}^* + 2u_n\tilde{\phi}^{*2}\tilde{\phi} + \\
&\qquad\quad + \frac{1}{a^2}v_n\frac{m^*}{g^*} + w_n\left[2\frac{m^*}{g^*}\tilde{\phi}^*\tilde{\phi} + \frac{m}{g}\tilde{\phi}^{*2}\right] + 2x_n\frac{m^{*2}}{g^{*2}}\tilde{\phi}\Big)\sqrt{2}K^TCP_+\tilde{\chi} + \\
&\qquad + y_n(2g\tilde{F}\tilde{\phi}\sqrt{2}K^TCP_+\tilde{\chi} + g\tilde{\phi}^2\sqrt{2}K^TC\gamma_\mu\partial_\mu P_+\tilde{\chi}) + \\
&\qquad + z_n(m\tilde{F}\sqrt{2}K^TCP_+\tilde{\chi} + m\tilde{\phi}\sqrt{2}K^TC\gamma_\mu\partial_\mu P_+\tilde{\chi}) + \text{c.c.}\Big\} + \\
&+ O(K^2, L, \ldots) \,.
\end{aligned} \tag{3.51}
$$

It is easy to verify that these are, modulo supersymmetric combinations, most general local terms whose mass dimension ≤ 4 that are consistent with the $\mathrm{U}(1)_R$ and U(1) global symmetries (the U(1) charges of the external source $P_\pm K$ is $(\mp 3, \pm 1)$). This is an expected result from an experience in continuum theory but is not entirely trivial, because the term which breaks supersymmetry due to a violation of the Leibniz rule (3.9) is peculiar to lattice theory and cannot be treated in a framework of continuum theory.

To remedy the above breaking of supersymmetry, we subtract eq. (3.51) from $S_{\text{tot.}}$ after transcribing it as local terms in lattice theory by substitutions

$$
\int \mathrm{d}^4x \to a^4\sum_x , \qquad \partial_\mu^2 \to D_1^2 , \qquad \gamma_\mu\partial_\mu \to D_1 \,. \tag{3.52}
$$

This is the improvement step; we have to add 9 combinations of local terms to $S_{\text{tot.}}$ for Γ_n to have a supersymmetric continuum limit. Then a further supersymmetric renormalization (3.20) will make Γ_n finite. Although this procedure may be applied in principle, the number of required local terms for the improvement is too many for any practical application of the present model.

We can understand why the situation in the one-loop level was so simple by considering the case in which the first term in eq. (3.48) is 7/2 instated of 9/2. Repeating a similar analysis as above, as a possible form of $\lim_{a\to 0}\sum_x \frac{\partial}{\partial K_\alpha}\Gamma_1\epsilon_\alpha|_{K=L=\cdots=0}$, we obtain

$$-g^* g\int d^4x\,\left\{r_1\tilde{F}^*\sqrt{2}\epsilon^T C\gamma_\mu\partial_\mu P_+\tilde{\chi}+s_1\partial_\mu^2\tilde{\phi}^*\sqrt{2}\epsilon^T CP_+\tilde{\chi}\right\} \tag{3.53}$$

plus the complex conjugate. This has a much simpler structure than eq. (3.50). We in fact found the term with r_1 in the explicit one-loop calculation (3.43). We also observe that a general argument does not prohibit a non-zero s_1; thus $s_1=0$ in the explicit one-loop calculation (3.43) is accidental.

3.6 Exact non-linear fermionic symmetry

Finally, we comment on an exact non-linear fermionic symmetry in this system which corresponds to the symmetry studied in ref. [21]. We note that the lattice Ward-Takahashi identity (3.6) (after setting $K=L=M=\cdots=0$) may be written as

$$\begin{aligned}\Big\langle -a^4\sum_x\{&J_\chi\delta_\epsilon\chi+J_\phi\delta_\epsilon\phi+J_{\phi^*}\delta_\epsilon\phi^*+J_F\delta_\epsilon F+J_{F^*}\delta_\epsilon F^*+\\&+J_X\delta_\epsilon X+J_\Phi\delta_\epsilon\Phi+J_{\Phi^*}\delta_\epsilon\Phi^*+J_{\mathcal{F}}\delta_\epsilon\mathcal{F}+J_{\mathcal{F}^*}\delta_\epsilon\mathcal{F}^*\}+\\&+\sum_x\left\{S\frac{\overleftarrow{\partial}}{\partial\chi}R\epsilon+S\frac{\overleftarrow{\partial}}{\partial X}\mathcal{R}\epsilon\right\}\Big\rangle=0\,,\end{aligned} \tag{3.54}$$

where

$$\begin{aligned}R&=\left[D+mP_++m^*P_-+2(g\tilde{\phi}P_++g^*\tilde{\phi}^*P_-)\left\{1-\frac{1}{2}a(D+mP_++m^*P_-)\right\}\right]^{-1}\Delta L\\ \mathcal{R}&=-\frac{1}{2}a(D+mP_++m^*P_-)R\,.\end{aligned} \tag{3.55}$$

However, noting the Schwinger-Dyson equation

$$\left\langle\sum_x S\frac{\overleftarrow{\partial}}{\partial\varphi}\delta\varphi+a^4\sum_x J_\varphi\delta\varphi\right\rangle=0\,, \tag{3.56}$$

where φ and $\delta\varphi$ represent a generic field and its *arbitrary* variation, the identity (3.54) can be cast into the form

$$\begin{aligned}\Big\langle -a^4\sum_x\{&J_\chi\delta'_\epsilon\chi+J_\phi\delta'_\epsilon\phi+J_{\phi^*}\delta'_\epsilon\phi^*+J_F\delta'_\epsilon F+J_{F^*}\delta'_\epsilon F^*+\\&+J_X\delta'_\epsilon X+J_\Phi\delta'_\epsilon\Phi+J_{\Phi^*}\delta'_\epsilon\Phi^*+J_{\mathcal{F}}\delta'_\epsilon\mathcal{F}+J_{\mathcal{F}^*}\delta'_\epsilon\mathcal{F}^*\}\Big\rangle=0\,,\end{aligned} \tag{3.57}$$

where δ'_ϵ is a transformation modified by amount of R and $\mathcal{R}$:

$$\begin{aligned}\delta'_\epsilon\chi&=-\sqrt{2}P_+(D_1\phi+F)\epsilon-\sqrt{2}P_-(D_1\phi^*+F^*)\epsilon+R\epsilon\,,\\ \delta'_\epsilon X&=-\sqrt{2}P_+(D_1\Phi+\mathcal{F})\epsilon-\sqrt{2}P_-(D_1\Phi^*+\mathcal{F}^*)\epsilon+\mathcal{R}\epsilon\,,\\ \delta'_\epsilon&=\delta_\epsilon\qquad\text{on other fields}\,.\end{aligned} \tag{3.58}$$

Obviously, the above form of the identity (3.57) can be regarded as a Ward-Takahashi identity associated to an *exact* symmetry δ'_ϵ of the action S. The transformation δ'_ϵ when acting on tilded variables (i.e., in the context of the Fujikawa-Ishibashi formulation) is nothing but the exact non-linear transformation studied in ref. [21]. However, as we have demonstrated, Ward-Takahashi identities in both pictures, one is based on δ_ϵ (eq. (3.54)) and another is based on δ'_ϵ (eq. (3.57)), have identical contents.[16] The presence of this exact symmetry itself does not imply the lattice formulation is "better" in any sense. Despite this exact symmetry, an adjustment of parameters is needed to obtain a supersymmetric continuum limit, as we have discussed in this paper.

4. Conclusion

In this paper, we formulated a lattice model for the $N = 1$ supersymmetric Wess-Zumino model in 4 dimensions. The $\mathrm{U}(1)_R$ symmetry is manifest even on a lattice with a use of Ginsparg-Wilson fermions. Although our formulation is perturbatively equivalent to the Fujikawa-Ishibashi formulation, we could avoid a singular nature of the latter formulation by introducing an auxiliary chiral supermultiplet on a lattice. We also analyzed radiative breaking of the supersymmetric Ward-Takahashi identity. The situation in the one-loop order is rather simple while the improvement through higher orders will be much more involved due to a peculiarity of the supersymmetry breaking term. In particular, in higher orders, we cannot avoid an adjustment of mass parameters of scalar fields which are quadratically diverging. In this aspect, the situation is not quite better than for the formulation based on the Wilson fermion. Clearly, a much clever idea is needed to achieve a lattice formulation of the $N = 1$ Wess-Zumino model which avoids this too much adjustment.

Note added in proof. In an illustrative work [39], Giedt, Koniuk, Poppitz and Yavin studied a fine-tuning problem in a naively discretized supersymmetric quantum mechanics by utilizing the Reisz theorem. We thank Joel Giedt for calling their work to our attention.

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A. Cosmological term at the one-loop level

The cosmological term (the vacuum energy) at the one-loop level is given by a logarithm of the partition function with $g = 0$. For a correct counting of degrees of freedom, it is

[16] Actually, this is the starting point of the argument of ref. [7].

helpful to introduce real bosonic variables by

$$\begin{aligned}
\phi &\to \frac{(A+iB)}{\sqrt{2}}\,, \qquad & F &\to \frac{(F-iG)}{\sqrt{2}}\,, \\
\Phi &\to \frac{(\alpha+i\beta)}{\sqrt{2}}\,, \qquad & \mathcal{F} &\to \frac{(\mathcal{F}-i\mathcal{G})}{\sqrt{2}}\,.
\end{aligned} \tag{A.1}$$

Then the free part of action S_0 is represented as

$$\begin{aligned}
S_0 = a^4 \sum_x (A\ B\ F\ G\ \alpha\ \beta\ \mathcal{F}\ \mathcal{G}) M_{\mathrm{B}} \begin{pmatrix} A \\ B \\ F \\ G \\ \alpha \\ \beta \\ \mathcal{F} \\ \mathcal{G} \end{pmatrix} + \\
+ a^4 \sum_x ((\chi P_+)^T\ (\chi P_-)^T\ (XP_+)^T\ (XP_-)^T) M_{\mathrm{F}} \begin{pmatrix} P_+\chi \\ P_-\chi \\ P_+X \\ P_-X \end{pmatrix},
\end{aligned} \tag{A.2}$$

where matrices M_{B} and M_{F} are given by

$$\begin{aligned}
M_{\mathrm{B}} &= \\
&= \frac{1}{2} \begin{pmatrix}
D_1^2 & 0 & D_2+R & I & 0 & 0 & R & I \\
0 & D_1^2 & -I & D_2+R & 0 & 0 & -I & R \\
D_2+R & -I & 1 & 0 & R & -I & 0 & 0 \\
I & D_2+R & 0 & 1 & I & R & 0 & 0 \\
0 & 0 & R & I & 0 & 0 & -\frac{2}{a}+R & I \\
0 & 0 & -I & R & 0 & 0 & -I & -\frac{2}{a}+R \\
R & -I & 0 & 0 & -\frac{2}{a}+R & -I & 0 & 0 \\
I & R & 0 & 0 & I & -\frac{2}{a}+R & 0 & 0
\end{pmatrix},
\end{aligned} \tag{A.3}$$

where $R = \mathrm{Re}\, m$ and $I = \mathrm{Im}\, m$ and

$$M_{\mathrm{F}} = \frac{1}{2} C \begin{pmatrix}
D_2+m & D_1 & m & 0 \\
D_1 & D_2+m^* & 0 & m^* \\
m & 0 & -\frac{2}{a}+m & 0 \\
0 & m^* & 0 & -\frac{2}{a}+m^*
\end{pmatrix}. \tag{A.4}$$

By noting the relation (2.11), we then find that gaussian integrations over bosonic variables give rise to the following contribution to the partition function

$$\det{}^{-1} \left\{ \frac{-1}{4a^2} \left[\frac{2}{a} D_2 + (1 - \frac{1}{2} a D_2) m^* m \right] \right\}. \tag{A.5}$$

On the other hand, integrations over fermionic variables give rise to

$$\det C \det \left\{ \frac{1}{4a} \left[\frac{2}{a} D_2 + (1 - \frac{1}{2} a D_2) m^* m \right] \right\}, \tag{A.6}$$

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where we have taken into account the fact that χ and X are 4 component spinors. These contributions from bosonic and fermionic variables are cancelled to each other, leaving a constant which may be normalized to unity. Therefore, the cosmological term is not generated by one-loop radiative corrections *even for* $a \neq 0$, as expected from the exact supersymmetry of the free action S_0.

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